PROJECTED WRITTEN NOTES FROM THE M325K LECTURE ON TUESDAY, FEBRUARY 27, 2024, on

MORE ON STRONG MATHEMATICAL INDUCTION and ON THE WELL-ORDERING PRINCIPLE

CLASS #13

A Strong math'l Induction EXAMPLE

Sec. 5.4 #5

Suppose that $e_0, e_1, e_2, \ldots$ is a sequence defined as follows:*

$e_0 = 12, \quad e_1 = 29,$

$e_t = 5e_{t-1} - 6e_{t-2}$ for all integers $t \geq 2$

To Prove: For all integers $n \geq 0$,

$$e_n = 5 \times 3^n + 7 \times 2^n.$$

Proof: [By Strong Mathematical Induction]

Let $n = 0$. $e_n = e_0 = 12$.

$5 \times 3^n + 7 \times 2^n = 5 \times 3^0 + 7 \times 2^0 = 5 + 7 = 12$

Let $n = 1$. $e_1 = 29$

$5 \times 3^n + 7 \times 2^n = 5 \times 3^1 + 7 \times 2^1 = 15 + 14 = 29$

So, For $n = 0$ and $n = 1$, $e_n = 5 \times 3^n + 7 \times 2^n$.

[END of BASIS STEP].

[INDUCTIVE STEP]

Let k be any integer such that $k \geq 1$.

Suppose that, for every integer m such that $0 \leq m \leq k$, $e_m = 5 \times 3^m + 7 \times 2^m$. [Ind Hyp]

[NTS: $e_{k+1} = 5 \cdot 3^{(k+1)} + 7 \times 2^{(k+1)}$]

Since $k \geq 1$, $k+1 \geq 2$.

Since $k+1 \geq 2$, $e_{k+1} = 5e_k - 6e_{k-1}$, from the formula in the sequence definition.

[Apply the Induction Hypothesis to e_k and e_{k-1}]

Since $k \geq 1$, $k-1 \geq 0$.

$\therefore 0 \leq k-1 \leq k$ and $0 \leq k \leq k$.

By the Ind. Hyp., $e_k = 5 \times 3^k + 7 \times 2^k$

and, $e_{k-1} = 5 \times 3^{(k-1)} + 7 \cdot 2^{k-1}$

By Subst., $e_{k+1} = 5(5 \times 3^k + 7 \times 2^k) - 6(5 \times 3^{(k-1)} + 7 \times 2^{k-1})$

$= 25 \times 3^k + 35 \times 2^k - 30 \cdot 3^{(k-1)} - 42 \cdot 2^{k-1}$

$= 25 \times 3^k + 35 \times 2^k - 10 \times 3^k - 21 \times 2^k$

$= 15 \times 3^k + 14 \times 2^k$

$\boxed{e_{k+1} = 5 \times 3^{k+1} + 7 \cdot 2^{k+1}}$

$\therefore$ For all integers $k \geq 1$, if $e_k = 5 \times 3^k + 7 \times 2^k$

then $e_{k+1} = 5 \times 3^{k+1} + 7 \times 2^{k+1}$, by direct proof.

[END of the Inductive Step]

$\therefore$ For all integers $n \geq 0$,

$e_n = 5 \times 3^n + 7 \times 2^n$ by Strong Math'l Induction.

QED

The Well-Ordering Principle for the Integers

Some proofs in mathematics use a property of the set of integers called the Well-Ordering Principle.

Definition of the Well-Ordering Principle for the Integers

Every non-empty set of integers
in which every element is greater than or equal to some fixed integer
has a least element.

More formally, specifying Condition (1) and Condition (2):

Let set S be a set of integers such that:

1) There is at least one integer element in S, and

2) There exists an integer L such that every element in set S is greater than or equal to L.

Then, by the Well-Ordering Principle, S has a least element m.

Note: The integer L (thought of a "Lower Bound" of integers in S) is most likely not in the set S.

And, the choices of which integer L might be is never unique:
many different integers can be chosen as L to serve as a lower bound for set S.
For example, if $L = 2$ works, then $L = 1$, $L = 0$, $L = -1$, etc., all will also work.

Example 1: Verify that the Well-Ordering Principle can be applied to set S,

where set $S = \{$ All integers x such that $24 = xy$ for some integer $y\}$.

1) [Show that there is at least one integer element in set S.]

$24 = 12 \times 2 = xy$ where $x = 12$ and $y = 2$.
Thus, $x = 12$ is in set S, so there is at least one integer element in S, so S is a non-empty set.

2) [Show that there exists an integer L such that, for all x in S, $x \geq L$.]

By definition of S, every integer element in S is a divisor of 24.
The divisors of 24 range between -24 and $+24$, so every divisor of 24 is greater than or equal to -25. Thus, every element in set S is greater than or equal to -25.

$\therefore$ Set S satisfies Condition (1) and Condition (2) of the Well-Ordering Principle of the Integers.
$\therefore$ By the Well-Ordering Principle of the Integers, set S has a least element, m.

Regarding the step of locating an integer L to serve as a "Lower Bound" of the elements in set S, sometimes the definition of S itself says, for instance, that set S is the set of all integers n, $n \geq 0$, such that n has such-and-such property. In that case, verifying the existence of a "Lower Bound" for the elements of S is accomplished just by saying, "By definition of set S, every integer in S is greater than or equal to 0."

An EXAMPLE PROOF USING THE WELL-ORDERING PRINCIPLE. (W.O.P.)

<u>To Prove</u>: For all integers $n \geq 0$, $7^n - 1$ is divisible by 6.

<u>Proof</u>: Suppose, by way of contradiction, that there exists an integer $N \geq 0$ such that $7^N - 1$ is not divisible by 6.

Let $S = \{$ all integers t such that $t \geq 0$ and $7^t - 1$ is not divisible by 6. $\}$

By supposition, $N \geq 0$ and $7^N - 1$ is not divisible by 6, So, N is in Set S and set S is not empty.

Condition 1 of the W.O.P. is satisfied.

By definition of set S, for every integer t in Set S, $t \geq 0$.
Condition 2 of the W.O.P. is satisfied.

By the Well-Ordering Principle, set S has a least element, m.

Note that $m \geq 0$ and $7^m - 1$ is not divisible by 6.

[We show that $m \geq 1$ by showing that $m \neq 0$.]

$7^0 - 1 = 0 = 0 \times 6$, so $7^0 - 1$ is divisible by 6, but $7^m - 1$ is not divisible by 6.

So, $m \neq 0$. $\therefore m \geq 1$ $\therefore m - 1 \geq 0$.

[We show that $7^{m-1}-1$ is divisible by 6.]

Suppose, by way of contradiction, that $7^{m-1}-1$ is not divisible by 6. Recall that $m-1 \geq 0$.

$\therefore$ Integer $m-1$ is in set S and $m-1 < m$, which contradicts the fact that m is the least element in S.

$\therefore$ $7^{m-1}-1$ is divisible by 6, by proof-by-contradiction.

[We show that $7^{m}-1$ is divisible by 6].

Since $7^{m-1}-1$ is divisible by 6, there exists an integer k such that $7^{m-1}-1 = 6k$.

$\therefore$ $7^{m-1} = 6k+1$ by R.O.A.

$\therefore$ $7^{m} = 42k+7 = 6\times(7k)+6+1$

$\therefore$ $7^{m}-1 = 6(7k+1)$ and $7k+1$ is an integer.

$\therefore$ $7^{m}-1$ is divisible by 6, which contradicts the fact that $7^{m}-1$ is not divisible by 6.

$\therefore$ For all integers $n \geq 0$, $7^{n}-1$ is divisible by 6, by proof-by-contradiction.

Q.E.D.

A SECOND EXAMPLE PROOF USING the WELL-ORDERING PRINCIPLE (Sec 5.3, #18)

To Prove: $5^n + 9 < 6^n$ for all integers $n \geq 2$.

Proof: Suppose, by way of contradiction, that there exists an integer $N \geq 2$ such that $5^N + 9 \geq 6^N$.

Let set S = { all integers t such that $t \geq 2$ and $5^t + 9 \geq 6^t$ }

By supposition, $N \geq 2$ and $5^N + 9 \geq 6^N$.

So, N is in Set S, and so, set S is not empty.

$\therefore$ Condition 1 of the Well-Ordering Principle is satisfied.

By definition of Set S, $t \geq 2$ for every integer t in S.

$\therefore$ Condition 2 of the Well-Ordering Principle is satisfied.

$\therefore$ By the Well-Ordering Principle, set S has a least element, m.

$\therefore$ $m \geq 2$ and $5^m + 9 \geq 6^m$ since m is in S.

Also, since m is the least integer in Set S, if integer $k < m$, then k is not in Set S.

[Proof Continues on the next page.]

[We show that $m \geq 3$ by showing that $m \neq 2$]

$5^2+9=34$, and $6^2=36$ and $34<36$.

$\therefore$ When $n=2$, $5^n+9 \leq 6^n$. That is, $5^2+9<6^2$.

But, since m is in Set S, $5^m+9 \geq 6^m$.

So, $m \neq 2$. $\therefore m \geq 3$

$\therefore m-1 \geq 2$

[We show that $5^{m-1}+9 < 6^{m-1}$]

Suppose, by way of contradiction, that $5^{m-1}+9 \geq 6^{m-1}$.

Since $m-1 \geq 2$ and $5^{m-1}+9 \geq 6^{m-1}$, $m-1$ is in Set S.

But $m-1<m$ and $m-1$ is in Set S, which contradicts the fact that m is the least integer element in Set S.

$\therefore 5^{m-1}+9 < 6^{m-1}$, by proof-by-contradiction.

$\therefore 5(5^{m-1}+9) < 5(6^{m-1}) < 6(6^{m-1}) = 6^m$

$\therefore 5^m+45 < 6^m$ [Subtract 36 from both sides]

$\therefore 5^m+9 < 6^m-36 < 6^m$

$\therefore 5^m+9<6^m$, but $5^m+9 \geq 6^m$, a contradiction.

[So, N never existed at the start!]

$\therefore$ For all integers $n \geq 2$,

$5^n+9<6^n$, by proof-by-contradiction.

QED